# BOHR DENSITY OF SIMPLE LINEAR GROUP ORBITS

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ABSTRACT. We show that any nonzero orbit under a noncompact, simple, irreducible linear group is dense in the Bohr compactification of the ambient space.

## 1. INTRODUCTION

Let G be a noncompact simple real Lie group and V a finite-dimensional, simple real G-module. Write $V^*$ for the dual module and $bV$ for the Bohr compactification of V, i.e. $bV$ is the Pontrjagin dual of the additive group $V^*$ made discrete. The aim of this note is to prove the following result, which was conjectured in [Z96, p. 45]:

**Theorem.** *Every nonzero* G*-orbit in* V *is dense in* $bV$.

We note that an analogous result holds for G nilpotent and V a module of unipotent type [B72, p. 6], namely, each orbit $Gv$ is dense in the Bohr compactification of its affine hull [Z93]. In contrast we observe that the orbits of $\mathbf{R}$ acting on $\mathbf{R}^2$ by $\exp\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ are *not* dense in $b\mathbf{R}^2$.

## 2. PROOF OF THE THEOREM

As observed by Katznelson [K73a] (see also [G79, §7.6]), it is enough to show that Haar measure on $bV$ is the weak$*$ limit of probability measures $\mu_T$ concentrated on the orbit under consideration; or equivalently [B74], that the Fourier transforms of the $\mu_T$ tend pointwise to the characteristic function of $0 \in V^*$.

To construct such $\mu_T$, we assume without loss of generality that the action of G on V is effective, so that we may regard $G \subset GL(V)$. Let K be maximal compact in G, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subalgebra, $C \subset \mathfrak{a}^*$ a Weyl chamber, $P \subset \mathfrak{a}$ the dual positive cone, and H an interior point of P, so that $\langle \nu, H \rangle$ is positive for all nonzero $\nu \in C$. We fix a nonzero $v \in V$ and let $\mu_T$ denote the image of Haar $\times$ (Lebesgue/T) $\times$ Haar

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under the composed map

$$\begin{array}{rcccl} \mathrm{K}\times[0,\mathrm{T}]\times\mathrm{K} & \longrightarrow & \mathrm{G}v & \longrightarrow & b\mathrm{V} \\ (k,t,k') & \longmapsto & k\exp(t\mathrm{H})k'v & & \\ & & w & \longmapsto & \mathrm{e}^{\mathrm{i}\langle\cdot,w\rangle}. \end{array}$$

Here $\exp:\mathfrak{a}\to\mathrm{A}$ is the usual matrix exponential with inverse $\log:\mathrm{A}\to\mathfrak{a}$, and the brackets $\langle\cdot,\cdot\rangle$ denote both pairings, $\mathfrak{a}^*\times\mathfrak{a}\to\mathbf{R}$ and $\mathrm{V}^*\times\mathrm{V}\to\mathbf{R}$. There remains to show that as $\mathrm{T}\to\infty$ we have, for every nonzero $u\in\mathrm{V}^*$,

$$\int_{\mathrm{K}\times\mathrm{K}} dk\,dk'\,\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\langle u,k\exp(t\mathrm{H})k'v\rangle}\,dt\to 0. \tag{$*$}$$

To this end, let

$$\mathrm{F}_{kk'}(t)=\langle u,k\exp(t\mathrm{H})k'v\rangle$$

denote the exponent in ($*$). It is well known (see for example [**K73b**, Prop. 2.4 and proof of Prop. 3.4]) that $\mathfrak{a}$ acts diagonalizably (over $\mathbf{R}$) on V. Thus, letting $\mathrm{E}_\nu$ be the projector of V onto the weight $\nu$ eigenspace of $\mathfrak{a}$, we can write

$$\mathrm{F}_{kk'}(t)=\sum_{\nu\in\mathfrak{a}^*}\langle u,k\mathrm{E}_\nu k'v\rangle \mathrm{e}^{\langle\nu,\mathrm{H}\rangle t}.$$

Now we claim that there are nonzero $\nu$ such that the coefficient $f_\nu(k,k')=\langle u,k\mathrm{E}_\nu k'v\rangle$ is not identically zero on $\mathrm{K}\times\mathrm{K}$. (Then $f_\nu$, being analytic, will be nonzero *almost everywhere*.) Indeed, suppose otherwise. Then, writing any $g\in\mathrm{G}$ in the form $kak'$ (KAK decomposition [**K02**]), we would have

$$\langle u,gv\rangle=\sum_{\nu\in\mathfrak{a}^*}\langle u,k\mathrm{E}_\nu k'v\rangle \mathrm{e}^{\langle\nu,\log(a)\rangle}=\langle u,k\mathrm{E}_0k'v\rangle.$$

In particular the matrix coefficient $\langle u,gv\rangle$ would be bounded, and hence so would be all matrix coefficients (since they are linear combinations of translates of this one, since $u$ and $v$ are cyclic, since V and $\mathrm{V}^*$ are simple); and this would contradict the noncompactness of $\mathrm{G}\subset\mathrm{GL(V)}$.

Thus, we may pick a $\nu_0$ such that $f_{\nu_0}$ is not identically zero. Conjugating if necessary, we can assume that $\nu_0\in\mathrm{C}$, and choose it there so as to maximize $\langle\nu_0,\mathrm{H}\rangle$. Then our exponent writes:

$$\mathrm{F}_{kk'}(t)=\mathrm{e}^{\langle\nu_0,\mathrm{H}\rangle t}\left\{f_{\nu_0}(k,k')+\varepsilon_{kk'}(t)\right\}$$

where $\varepsilon_{kk'}(t)$ decays exponentially to zero as $t\to\infty$ for all $k,k'$. Now it is clear that for almost all $(k,k')$ there is a $\mathrm{T}_0$ beyond which $|d\mathrm{F}_{kk'}(t)/dt|>1$ and $d^2\mathrm{F}_{kk'}(t)/dt^2\neq 0$. By van der Corput's lemma [**S93**, p. 332] this implies that

$$\left|\int_{\mathrm{T}_0}^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\mathrm{F}_{kk'}(t)}\,dt\right|<3\qquad\forall\,\mathrm{T}.$$

Therefore we have $\lim_{\mathrm{T}\to\infty}\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\mathrm{F}_{kk'}(t)}\,dt=0$ for almost all $(k,k')$, whence the conclusion ($*$) by dominated convergence. This completes the proof.

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